

NOTES ON STEADY VORTEX MOTIONS OF CONTINUOUS MEDIUM*

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The general equation for variation of solenoidal vector field which is satisfied by the magnetic field intensity in magnetohydrodynamics and the vector of vortex barotropic flows of ideal fluid, with the equation of continuity taken into account, is reduced to the "Euler equation for the change of momentum". This note is used for investigating of the topology of steady barotropic flows of inviscid compressible fluid.

In the investigation of general properties of vortex lines an important part is played by the equation

$$\partial u / \partial t = \text{rot} (v \times u) \quad (1)$$

where u is some solenoidal vector field in a three-dimensional Euclidean space (its physical meaning depends on the specific formulation of the problem), and v is the velocity of particles of the continuous medium /1/.

The vector of magnetic intensity in a medium with infinite conductivity satisfies equation (1) /1/. The same equation defines the change of vector of vortex barotropic flows of ideal fluid (gas) in the potential mass force field.

It appears that Eq.(1) with the continuity equation taken into account can be represented in the form of Euler's equation for change of momentum

$$\partial w / \partial t = [v, w], \quad w = u / \rho \quad (2)$$

where the bracket $[,]$ is the commutator of vector fields and ρ is the density of the matter. In particular the steady case the fields v and w commute.

To prove (2) we calculate

$$\frac{\partial (u/\rho)}{\partial t} = \frac{1}{\rho} \text{rot} (v \times u) + \frac{u}{\rho^2} \text{div} (\rho v) \quad (3)$$

On the other hand

$$\left[v, \frac{u}{\rho} \right] = \frac{1}{\rho} [v, u] + \frac{u}{\rho^2} L_v \rho$$

where $L_v \rho$ is the derivative function of ρ along the vector fields v . Using the known identity of vector analysis, we obtain that

$$\left[v, \frac{u}{\rho} \right] = \frac{1}{\rho} \text{rot} (v \times u) + \frac{u}{\rho} \text{div} v + \frac{u}{\rho^2} L_v \rho \quad (4)$$

Since $L_v \rho + \rho \text{div} v = \text{div} (\rho v)$, then from (3) and (4) follows (2).

The steady motion of continuous medium with vector field u shall be called vortex flow, if $u \times v \neq 0$. To each point x in the region of flow it is natural to put the corresponding plane $\pi(x)$ that is generated by the linear combinations of independent vectors $u(x)$ and $v(x)$. Since the fields v and u/ρ commute, hence the distribution of planes $\pi(x)$ is involutory /2/. In particular, through each point x passes a unique integral surface M_x of that distribution, which touches vectors u and v . Generally, the surfaces M_x can be immersed in the region of flow in a very complex manner; generally, they are not closed.

The simplest description of motion of the continuous medium particles is over the compact integral surfaces. Let M be a compact surface without an edge. Since the touching fields v and w commute and are linearly independent, M is diffeomorphic to the two-dimensional torus T^2 , and in some angular coordinates $\varphi_1, \varphi_2 \bmod 2\pi$ the differential equations $\dot{x} = v(x)$ and $\dot{x} = w(x)$ on T^2 are of the form /3/

$$\dot{\varphi}_1 = \omega_1, \quad \dot{\varphi}_2 = \omega_2 \quad \text{and} \quad \varphi_1 = \Omega_1, \quad \varphi_2 = \Omega_2 \quad (\omega, \Omega = \text{const})$$

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Consequently, in the compact case the streamlines are either closed (and then the periods of particles circulation over different closed trajectories coincide), or they are everywhere compact on M .

In φ_1, φ_2 coordinates the equation $x' = u(x) = \rho(x) w(x)$ is evidently reduced to the form $\varphi_i' = \Omega_i \rho$ and the integral lines of the vector field u (in magnetohydrodynamics these are the force lines of the magnetic field) are also straight lines. Consequently they are all closed or they wrap compactly M . However, when $\rho \neq \text{const}$ then, unlike the stream lines, the integral field lines u close in different time intervals.

If the touching fields v and w are full on M , then in the noncompact case M is diffeomorphic to cylinder $R^1 \times T^1$ or to surface R^2 , and in some coordinates on M the streamlines and the integral lines of the vector field w also straighten out as a whole.

The described construction finds the most meaningful application in the problem of barotropic flows of ideal fluid. In that case from the equation of motion in the Lamb form follows that all integral surfaces are the same as the closed levels of the Bernoulli integral $f = c$.

Theorem. Let us assume that the flow region is compact and bounded by a regular analytic surface, and the velocity field v is analytic and $v \times \text{rot } v \neq 0$. Then almost all connected Bernoulli surfaces $M = \{f = c\}$ (except possibly a finite number) are diffeomorphic or to two-dimensional torus or ring $[0, 1] \times T^1$. On each of the tori all streamlines are either closed or everywhere dense, and on each ring they are closed. The period of revolution of fluid particles over different closed trajectories lying on a single Bernoulli surface coincide.

In the case of incompressible fluid this statement was proved in /4/.

Proof. The Lamb equation implies that f is a nonconstant analytic function. Hence all $f = c$ surfaces (except, possibly, some finite number) are regular /5/. On these surfaces obviously $v \times \text{rot } v \neq 0$. When $M = \{f = c\}$ has no common points with the boundary, it remains to carry out the already described general construction, and the case when M intersects the flow region boundary is considered exactly as in /4/.

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